Fast Numerical Solution of the Plasma Response Matrix for Real-time Ideal MHD Control

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***Abstract*—To help effectuate near real-time feedback control of ideal MHD instabilities in tokamak geometries, a parallelized version of A.H. Glasser's DCON (Direct Criterion of Newcomb) code is developed. To motivate the numerical implementation, we first solve DCON's δW formulation with a Hamilton-Jacobi theory, elucidating analytical and numerical features of the ideal MHD stability problem. The plasma response matrix is demonstrated to be the solution of an ideal MHD Riccati equation. We then describe our adaptation of DCON with numerical methods natural to solutions of the Riccati equation, parallelizing it to enable its operation in near real-time. We replace DCON's serial integration of perturbed modes—which satisfy a singular Euler-Lagrange equation—with a domain-decomposed integration of state transition matrices. Output is shown to match results from DCON with high accuracy, and with computation time < 1s. Such computational speed may enable active feedback ideal MHD stability control, especially in plasmas whose ideal MHD equilibria evolve with inductive timescale $\tau \gtrsim 1$s—as in ITER. Further potential applications of this theory are discussed.**

I. Introduction

Active feedback control of plasma stability is essential to achieving high performance in advanced tokamaks. Experiments with control systems for tearing modes, ELMs, and divertor fluxes have demonstrated success in mitigating such instabilities in tokamaks.[6][7][9] However, state-of-the-art active feedback methods are as yet limited to controlling instabilities *after* they are observed—(so-called 'catch-and-subdue' strategies). These methods therefore limit the real-time capability of stability control systems, and do little to address vulnerability to spontaneous, rapidly growing unstable modes. Ideal MHD instabilities in particular, which are experimentally observed to grow at timescales between Alfvénic and resistive time ($\tau_A < \tau_{\mathrm{MHD}} < \tau_R$), remain unaddressed by existing control systems.

We note that while the growth of unstable MHD modes is quite rapid, the evolution of *stable* tokamak equilbria is considerably slower. Magnetic field geometry evolves on an inductive timescale $\tau_{L/R} \gtrsim 1\mathrm{s} \gg \tau_R \gg \tau_A$. This separation of timescales creates a window of opportunity to prevent plasma instabilities *before* they start—that is, provided we are able to quickly enough steer a tokamak away from unstable equilibria, (and thereby maintain its operation in stable equilibrium), at a timescale $\tau_{\mathrm{control}} < \tau_{L/R}$.

In this work, we demonstrate the viability of the analytical and computational aspects of a *pre-emptive* active feedback ideal MHD control system. Such a system would require (i) diagnosis and fitting of the plasma equilibrium; (ii) analysis of the stability characteristics of that equilibrium; and (iii) an active controller to steer the plasma away from its stability boundaries, all within a timescale appropriate to the magnetic equilibrium evolution. This paper treats only the second—the stability analysis—of these three control system components.

We shall demonstrate that A.H. Glasser's DCON code may be adapted for near real-time use. We recast DCON's serial integration of its Euler-Lagrange equations (ELEs) in a language more familiar to control theory, and discover that the code becomes naturally parallelizable.

The rest of this paper is organized as follows: Section II demonstrates the equivalence of DCON's Lagrangian variational problem to a matrix Riccati differential equation (MRDE), emphasizing a physical intuition for the plasma response matrix. Section III describes numerical advantages of viewing DCON's integration as a control theoretic MRDE problem, particularly in the use of state transition matrices, which are commonly applied to such problems. Section IV presents methods undertaken to achieve high performance parallel code. And Section V summarizes these results and concludes.

II. Theory

We begin with a statement of this section's main result, which emphasizes the *plasma response matrix's* role as a bilinear form, mapping plasma edge perturbations to their effect on plasma energy.

<u>*Claim*</u>: The minimum bulk-fluid δW, for a stable, axisymmetric plasma equilibrium, is given by

$$\delta W[\mathbf{\Xi}, \mathbf{\Xi}^\dagger] = \mathbf{\Xi}^\dagger(1)\mathsf{P}(1)\mathbf{\Xi}(1) \quad (1)$$

where $\psi = 1$ indicates the plasma edge, $\mathbf{\Xi} \in \mathbb{C}^M$ is a Fourier-decomposed magnetic perturbation of the edge flux surface (truncated at M poloidal mode numbers), and matrix P is the solution to the MRDE

$$\mathsf{P}' = \mathsf{G} - \big[\mathsf{P} - \mathsf{K}^\dagger\big]\mathsf{F}^{-1}\big[\mathsf{P} - \mathsf{K}\big]. \quad (2)$$

(Note, the prime denotes a derivative with respect to ψ, and DCON's ψ-dependent matrices $\{\mathsf{F} = \mathsf{F}^\dagger, \mathsf{G} = \mathsf{G}^\dagger$, and $\mathsf{K} \neq \mathsf{K}^\dagger\} \in \mathbb{C}^{M\times M}$ describe the coupling between poloidal mode perturbations and the magnetic equilibrium, and the resulting effect on the plasma's energy—see a description of their calculation in [2].)

We demonstrate this claim in the following sections by (A) recalling DCON's δW Lagrangian formulation from [2]; (B) Legendre transforming to a Hamiltonian setting; (C) applying Hamilton-Jacobi theory; and (D) demonstrating the applicability of the Riccati formulation to

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DCON's axisymmetric ideal MHD model.

A. Revisiting the DCON Lagrangian

We begin by restating Eq. 19 of [2], which computes the change in energy δW resulting from the least stable magnetic perturbations of an axisymmetric toroidal plasma:

$$\delta W[\boldsymbol{\Xi},\boldsymbol{\Xi}^\dagger] \equiv \int_0^1 L(\boldsymbol{\Xi},\boldsymbol{\Xi}',\boldsymbol{\Xi}^\dagger,{\boldsymbol{\Xi}^\dagger}',\psi)\mathrm{d}\psi$$
$$= \frac{1}{2\mu_0}\int_0^1 \left({\boldsymbol{\Xi}^\dagger}'\mathsf{F}\boldsymbol{\Xi}' + {\boldsymbol{\Xi}^\dagger}'\mathsf{K}\boldsymbol{\Xi} + \boldsymbol{\Xi}^\dagger\mathsf{K}^\dagger\boldsymbol{\Xi}' + \boldsymbol{\Xi}^\dagger\mathsf{G}\boldsymbol{\Xi}\right)\mathrm{d}\psi. \tag{3}$$

Here $0 \le \psi \le 1$ is a flux surface label (radial) coordinate extending from the magnetic axis to the plasma edge, and plasma perturbation $\boldsymbol{\Xi} \in \mathbb{C}^M$ is a vector whose entries represent a radial displacement of the plasma along ψ, Fourier decomposed into M poloidal modes. Hermitian adjoints $\boldsymbol{\Xi}$ and $\boldsymbol{\Xi}^\dagger$ are taken to be independent dynamical variables due to their independent real and imaginary parts. We treat this integral for the perturbed energy as the action integral of DCON's Lagrangian, with coordinate ψ acting in lieu of a time parameter. Matrices $\{\mathsf{F}(\psi), \mathsf{G}(\psi),$ and $\mathsf{K}(\psi)\}$ are as described above.

Since a plasma is unstable to those perturbations which reduce its potential energy, we seek to characterize the perturbations $\boldsymbol{\Xi}$ which minimize δW. A necessary condition at any local extremum of δW is that its variational, $\delta(\delta W)$, vanishes for arbitrary variations to the perturbations, $\{\delta\boldsymbol{\Xi}, \delta\boldsymbol{\Xi}^\dagger\}$. It is in this sense that L in Eq. (3) represents the appropriate Lagrangian for stability analysis in our system.

B. The DCON Hamiltonian

Noting that L is Hermitian, and assuming our system is stable to all perturbations—(i.e., L is positive definite)—then L is convex, and we are free to Legendre transform this variational problem to its Hamiltonian formalism. (As our method is intended to pre-emptively analyze plasma stability *before* MHD instabilities arise, this is an entirely natural assumption to make.) Defining $\mathbf{q}_1 \equiv \boldsymbol{\Xi}$ and $\mathbf{q}_2^\dagger \equiv \boldsymbol{\Xi}^\dagger$, our system's canonical momenta (absorbing $1/2\mu_0$ for convenience) are:

$$\begin{aligned} \mathbf{p}_1^\dagger &\equiv L_{\mathbf{q}_1'} \equiv L_{\boldsymbol{\Xi}'} = \left({\boldsymbol{\Xi}^\dagger}'\mathsf{F} + \boldsymbol{\Xi}^\dagger\mathsf{K}^\dagger\right) \\ \mathbf{p}_2 &\equiv L_{{\mathbf{q}_2^\dagger}'} \equiv L_{{\boldsymbol{\Xi}^\dagger}'} = \left(\mathsf{F}\boldsymbol{\Xi}' + \mathsf{K}\boldsymbol{\Xi}\right). \end{aligned} \tag{4}$$

We thus derive the quadratic Hamiltonian:

$$\begin{aligned} H(\mathbf{q}_1,\mathbf{q}_2^\dagger,\mathbf{p}_2,\mathbf{p}_1^\dagger,\psi) &\equiv \left(\mathbf{p}_1^\dagger\mathbf{q}_1' + {\mathbf{q}_2^\dagger}'\mathbf{p}_2\right) - L(\mathbf{q}_1,\mathbf{q}_2^\dagger,\mathbf{p}_2,\mathbf{p}_1^\dagger) \\ &= \left({\boldsymbol{\Xi}^\dagger}'\mathsf{F} + \boldsymbol{\Xi}^\dagger\mathsf{K}^\dagger\right)\boldsymbol{\Xi}' + {\boldsymbol{\Xi}^\dagger}'\left(\mathsf{F}\boldsymbol{\Xi}' + \mathsf{K}\boldsymbol{\Xi}\right) \\ &\quad - \left({\boldsymbol{\Xi}^\dagger}'\mathsf{F}\boldsymbol{\Xi}' + {\boldsymbol{\Xi}^\dagger}'\mathsf{K}\boldsymbol{\Xi} + \boldsymbol{\Xi}^\dagger\mathsf{K}^\dagger\boldsymbol{\Xi}' + \boldsymbol{\Xi}^\dagger\mathsf{G}\boldsymbol{\Xi}\right) \\ &= \left({\boldsymbol{\Xi}^\dagger}'\mathsf{F}\boldsymbol{\Xi}' - \boldsymbol{\Xi}^\dagger\mathsf{G}\boldsymbol{\Xi}\right) \\ &= \begin{pmatrix}\mathbf{q}_2^\dagger & \mathbf{p}_1^\dagger\end{pmatrix}\begin{pmatrix}\mathsf{K}^\dagger\mathsf{F}^{-1}\mathsf{K} - \mathsf{G} & -\mathsf{K}^\dagger\mathsf{F}^{-1} \\ -\mathsf{F}^{-1}\mathsf{K} & \mathsf{F}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{q}_1 \\ \mathbf{p}_2\end{pmatrix}. \end{aligned} \tag{5}$$

When treating the system dynamically, we have been careful to separate $\mathbf{q}_1$ from $\mathbf{q}_2^\dagger$. Nevertheless, we note that $(\mathbf{q}_1)^\dagger = \mathbf{q}_2^\dagger$ and $(\mathbf{p}_1{}^\dagger)^\dagger = \mathbf{p}_2$. Throughout the remainder of this paper, we shall duly omit subscripts.

The original Lagrangian problem may therefore be reframed; we must find the plasma perturbations $\mathbf{q}$ (and their conjugate momenta $\mathbf{p}$) which extremize the action according to the Hamiltonian of Eq. (5):

$$\delta\int_0^1 \left(\mathbf{p}^\dagger\mathbf{q}' + {\mathbf{q}^\dagger}'\mathbf{p} - H\right)\mathrm{d}\psi = 0. \tag{6}$$

The perturbations which satisfy this variation are of course those obeying Hamilton's equations of motion (EOM)

$$\mathbf{q}' = \frac{\partial H}{\partial \mathbf{p}^\dagger} \quad \text{and} \quad \mathbf{p}' = -\frac{\partial H}{\partial \mathbf{q}^\dagger}, \tag{7}$$

which can be expressed compactly in the following $2M$ degree-of-freedom linear dynamical system:

$$\begin{pmatrix}\mathbf{q} \\ \mathbf{p}\end{pmatrix}' = \begin{pmatrix}-\mathsf{F}^{-1}\mathsf{K} & \mathsf{F}^{-1} \\ \mathsf{G} - \mathsf{K}^\dagger\mathsf{F}^{-1}\mathsf{K} & \mathsf{K}^\dagger\mathsf{F}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{q} \\ \mathbf{p}\end{pmatrix}. \tag{8}$$

The variation Eq. (6), after an integration by parts of the variationals of its first two terms, also yields an additional boundary term, which must vanish at extrema:

$$\mathbf{p}^\dagger\delta\mathbf{q}\Big|_0^1. \tag{9}$$

In an axisymmetric toroidal magnetic system, the radial perturbation at the magnetic axis, $\mathbf{q}_0 \equiv \boldsymbol{\Xi}(0)$, vanishes. (Consider, for example, that minimal-energy plasma perturbations satisfy $|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}|^2 = 0$.) Therefore, $\delta\mathbf{q}(0) = 0$ by assumption. At the the far plasma edge, however, the variation $\delta\mathbf{q}$ is arbitrary, and we therefore conclude

$$\mathbf{q}\Big|_{\psi=0} = \mathbf{p}\Big|_{\psi=1} = 0 \tag{10}$$

are appropriate boundaries conditions for the system solutions.

C. Hamilton-Jacobi Theory

We solve our system using a classical strategy of Hamiltonian theory, canonically transforming to the corresponding Hamilton-Jacobi problem (as described in [4]). In this approach, the problem is simplified by performing a canonical transformation that zeros the Hamiltonian everywhere. Recalling that a canonical transformation $(\mathbf{q},\mathbf{p};H)\mapsto(\mathbf{Q},\mathbf{P};K)$ must preserve the Lagrangian up to a total 'time' derivative

$$\mathbf{p}^\dagger\mathbf{q}'+\mathbf{q}^{\dagger\prime}\mathbf{p}-H=\mathbf{P}^\dagger\mathbf{Q}'+\mathbf{Q}^{\dagger\prime}\mathbf{P}-K+\frac{dF}{d\psi}, \tag{11}$$

we use a type-2 generating function, demanding that

$$F=F_2(\mathbf{q},\mathbf{q}^\dagger,\mathbf{P},\mathbf{P}^\dagger,\psi)-\mathbf{Q}^\dagger\mathbf{P}-\mathbf{P}^\dagger\mathbf{Q} \tag{12}$$

and setting $K=0$. This canonical transformation, substituted into Eq. (11), yields three equations (or five, including equivalent adjoints):

$$\begin{aligned}\mathbf{p}&=\frac{\partial S}{\partial\mathbf{q}^\dagger}\\ \mathbf{Q}&=\frac{\partial S}{\partial\mathbf{P}^\dagger}\\ 0&=H+\frac{\partial S}{\partial\psi}\end{aligned} \tag{13}$$

where $H=H(\mathbf{q},\mathbf{q}^\dagger,\mathbf{p},\mathbf{p}^\dagger,\psi)$ and S is *Hamilton's principal function*, a solution to $F_2=S(\mathbf{q},\mathbf{q}^\dagger,\mathbf{P},\mathbf{P}^\dagger,\psi)$, to be determined. Substituting Eqs. (13) into Eq. (5), we find that S must satisfy the following PDE:

$$\begin{aligned}S_\psi+\mathbf{q}^\dagger\left(\mathsf{K}^\dagger\mathsf{F}^{-1}\mathsf{K}-\mathsf{G}\right)\mathbf{q}-\mathbf{q}^\dagger\mathsf{K}^\dagger\mathsf{F}^{-1}S_{\mathbf{q}^\dagger}&\\ -S_{\mathbf{q}}^\dagger\mathsf{F}^{-1}\mathsf{K}\mathbf{q}+S_{\mathbf{q}}^\dagger\mathsf{F}^{-1}S_{\mathbf{q}^\dagger}&=0.\end{aligned} \tag{14}$$

Before solving for S, we note that $K=0$ implies *constant* $\mathbf{Q}\equiv\boldsymbol{\beta}$ and $\mathbf{P}\equiv\boldsymbol{\alpha}$, because $\mathbf{Q}'=\frac{\partial K}{\partial\mathbf{P}}=0=-\frac{\partial K}{\partial\mathbf{Q}}=\mathbf{P}'$. Substituting from Eq. (13), therefore, the total derivative of S is found to be:

$$\frac{dS}{d\psi}=-H+\left(\mathbf{p}^\dagger\mathbf{q}'+\mathbf{q}^{\dagger\prime}\mathbf{p}\right). \tag{15}$$

Upon comparison with Eq. (6), we note that we have recovered the well-known result of Hamilton-Jacobi theory:

$$S(\mathbf{q},\mathbf{q}^\dagger,\boldsymbol{\alpha},\boldsymbol{\alpha}^\dagger,\psi)=\int^{\psi}L\mathrm{d}\psi. \tag{16}$$

As a consequence S is seen to be the action of our system, and its importance is underscored by noting that

$$S\Big|_{\psi=1}=\int_0^1 L\mathrm{d}\psi=\delta W[\boldsymbol{\Xi},\boldsymbol{\Xi}^\dagger]. \tag{17}$$

We note that Eq. (14) represents a second order ODE system as a first order PDE. Given initial conditions, a solution S to this PDE would yield solutions for $\mathbf{q}$ (and thus $\mathbf{p}$) as well.

D. The Riccati Formulation

Given Eq. (13), it is natural to attempt a solution for S of the form:

$$S=\mathbf{q}^\dagger\mathbf{p}. \tag{18}$$

Let us assume (and we shall prove in a moment) that our dynamical variables obey the linear dependence

$$\mathbf{p}(\psi)=\mathsf{P}(\psi)\mathbf{q}(\psi) \tag{19}$$

for some matrix P. (Note matrix P—sans serif—is not to be confused with canonical coordinate vector $\mathbf{P}$.) Using the product rule to take the derivative of Eq. (19), and substituting from Eq. (8), we derive *DCON's Riccati formulation*:

$$\mathsf{P}'=\mathsf{G}-\left[\mathsf{P}-\mathsf{K}^\dagger\right]\mathsf{F}^{-1}\left[\mathsf{P}-\mathsf{K}\right]. \tag{20}$$

(We note by the self-adjointness of this ODE that P is everywhere Hermitian if it is anywhere Hermitian.) Given Eqs. (18) and (19), we therefore set

$$S(\mathbf{q},\mathbf{q}^\dagger,\psi)=\mathbf{q}^\dagger\mathsf{P}(\psi)\mathbf{q}. \tag{21}$$

Upon substitution into the PDE of Eq. (14), we find that the latter immediately reduces to the ODE derived in Eq. (20). The validity of the solution for S in Eq. (21) is thereby demonstrated.

We thus have only left to show that there exists a matrix $\mathsf{P}(\psi)$ satisfying Eq. (19). We do so by construction, finding P explicitly, as follows. We first concisely denote Eq. (8) as

$$\mathbf{x}'(\psi)=\mathsf{L}(\psi)\mathbf{x}(\psi). \tag{22}$$

We consider the fundamental matrix of solutions $\boldsymbol{\Phi}$ for this ODE, a $2M\times2M$ matrix whose columns form solutions spanning independent initial conditions of the system:

$$\begin{gathered}\boldsymbol{\Phi}=\begin{bmatrix}| & & |\\ \mathbf{x}_1 & \cdots & \mathbf{x}_{2M}\\ | & & |\end{bmatrix}\\ \boldsymbol{\Phi}'(\psi)=\mathsf{L}(\psi)\boldsymbol{\Phi}(\psi),\text{ where }\boldsymbol{\Phi}(0)=\mathbb{1}.\end{gathered} \tag{23}$$

Since $\boldsymbol{\Phi}_0\equiv\boldsymbol{\Phi}(0)=\mathbb{1}$ is nonsingular, $\mathbf{x}_0=\boldsymbol{\Phi}_0\mathbf{s}$ for some constant vector $\mathbf{s}$. But by ODE linearity, it must then hold true for all ψ that $\mathbf{x}(\psi)=\boldsymbol{\Phi}(\psi)\mathbf{s}$. Combining these relationships, we find a familiar result from linear ODE theory:

$$\mathbf{x}(\psi)=\boldsymbol{\Phi}(\psi)\left(\boldsymbol{\Phi}_0^{-1}\mathbf{x}_0\right)=\boldsymbol{\Phi}(\psi)\mathbf{x}_0. \tag{24}$$

In this way, $\boldsymbol{\Phi}(\psi)$ is to be regarded as the *state transition matrix* of the system, which maps ODE solutions forward in ψ. Thus we may write Eq. (24) as

$$\begin{pmatrix}\mathbf{q}(\psi)\\ \mathbf{p}(\psi)\end{pmatrix}=\begin{pmatrix}\boldsymbol{\Phi}_{\mathbf{qq}}(\psi) & \boldsymbol{\Phi}_{\mathbf{qp}}(\psi)\\ \boldsymbol{\Phi}_{\mathbf{pq}}(\psi) & \boldsymbol{\Phi}_{\mathbf{pp}}(\psi)\end{pmatrix}\begin{pmatrix}\mathbf{q}_0\\ \mathbf{p}_0\end{pmatrix}. \tag{25}$$

While this is enough to assert the desired linear dependence between $\mathbf{q}$ and $\mathbf{p}$, in our system this dependence can be simplified even further. As noted above—see Eq.(10)—the perturbation at the magnetic axis of our toroidal plasma satisfies $\mathbf{q}_0=0$, and thus:

$$\begin{pmatrix}\mathbf{q}(\psi)\\ \mathbf{p}(\psi)\end{pmatrix}=\begin{pmatrix}\boldsymbol{\Phi}_{\mathbf{qp}}(\psi)\mathbf{p}_0\\ \boldsymbol{\Phi}_{\mathbf{pp}}(\psi)\mathbf{p}_0\end{pmatrix}, \tag{26}$$

or, simply,

$$\mathbf{p}(\psi) = \Big(\mathbf{\Phi}_{\mathbf{pp}}(\psi)\mathbf{\Phi}_{\mathbf{qp}}^{-1}(\psi)\Big)\mathbf{q}(\psi). \tag{27}$$

This is in the desired form of Eq. (19), and thus provides a *solution* to the Riccati equation

$$\mathbf{P} = \mathbf{\Phi}_{\mathbf{pp}}\mathbf{\Phi}_{\mathbf{qp}}^{-1}. \tag{28}$$

E. Analysis: The Plasma Response Matrix

At the plasma edge, the Riccati solution

$$\mathbf{P}(1) = \mathbf{\Phi}_{\mathbf{pp}}(1)\mathbf{\Phi}_{\mathbf{qp}}^{-1}(1) \equiv \mathbf{W_P} \tag{29}$$

comprises the plasma response matrix. It may be regarded as the plasma permeability, or the 'conjugate momentum' created per unit of plasma displacement, (i.e., $\mathbf{W_P} \sim \mathbf{p}/\mathbf{q}$).

The previous section makes clear why $\mathbf{W_P}$ is appropriately viewed as the bilinear form which maps a perturbation $\mathbf{q}(1)$ at the plasma surface to its associated energetic 'cost', δW. This interpretation reemphasizes the importance of its eigenvalues to studies of plasma stability. In particular, the system's energy due to a perturbation is expressly given by:

$$\delta W[\mathbf{\Xi}, \mathbf{\Xi}^\dagger] = S\Big|_{\psi=1} = \mathbf{\Xi}^\dagger(1)\mathbf{P}(1)\mathbf{\Xi}(1) = \mathbf{\Xi}^\dagger(1)\mathbf{W_P}\mathbf{\Xi}(1). \tag{30}$$

This recovers a result originally presented in [2], (derived therein as a consequence of symmetrizing DCON's action), and proves our claim.□

III. Numerical Features of the Riccati Solution

Treating the fundamental matrix of solutions $\mathbf{\Phi}$ as a control theoretic state transition matrix—or *propagator*—highlights its usefulness in parallelizing the integration of DCON's ODE. In particular, as seen in Eq. (25), $\mathbf{\Phi}$ is an operator that maps solutions forward in ψ. As a result, any interval of the ODE Eq. (23) can be subdivided so that $\mathbf{\Phi}$ is comprised of its subpropagators:

$$\mathbf{x}(\psi_2) = \mathbf{\Phi}(\psi_2, \psi_0)\mathbf{x}(\psi_0) = \mathbf{\Phi}(\psi_2, \psi_1)\mathbf{\Phi}(\psi_1, \psi_0)\mathbf{x}(\psi_0), \tag{31}$$

where $\mathbf{\Phi}(\psi_i, \psi_i) = \mathbb{1}$ is the subinterval initial condition for $\forall i$. Integration of Eq. (23) across subdomains is thereby reduced to the multiplication of matrices that may themselves be independently calculated.

Furthermore, since the ultimate aim of our calculation is $\mathbf{W_P} = \mathbf{\Phi}_{\mathbf{pp}}(1)\mathbf{\Phi}_{\mathbf{qp}}^{-1}(1)$, we are free to transform the *cumulative subpropagator*

$$\mathbf{\Phi}_N \equiv \prod_{i=1}^{N} \mathbf{\Phi}_{i,i-1} \equiv \prod_{i=1}^{N} \mathbf{\Phi}(\psi_i, \psi_{i-1}) \tag{32}$$

$\forall\, N \geq 1$ by any right-multiplied linear operator of the form $\mathbf{R}_N = \begin{pmatrix} \mathbf{X}_N & \mathbb{0} \\ \mathbf{Y}_N & \mathbf{A}_N \end{pmatrix}$ with $\mathbf{A}_N$ nonsingular. For example, the product

$$\mathbf{\Phi}_{21}\left(\mathbf{\Phi}_{10}\mathbf{R}_1\right)\mathbf{R}_2 = \begin{pmatrix} \mathbf{\Phi}_{\mathbf{qq}} & \mathbf{\Phi}_{\mathbf{qp}} \\ \mathbf{\Phi}_{\mathbf{pq}} & \mathbf{\Phi}_{\mathbf{pp}} \end{pmatrix}_{21} \left(\begin{pmatrix} \mathbf{\Phi}_{\mathbf{qq}} & \mathbf{\Phi}_{\mathbf{qp}} \\ \mathbf{\Phi}_{\mathbf{pq}} & \mathbf{\Phi}_{\mathbf{pp}} \end{pmatrix}_{10} \mathbf{R}_1\right)\mathbf{R}_2 \tag{33}$$

leaves

$$\mathbf{W_P} = \left(\mathbf{\Phi}_{2\mathbf{qp}}\mathbf{A}_1\mathbf{A}_2\right)\left(\mathbf{\Phi}_{2\mathbf{pp}}\mathbf{A}_1\mathbf{A}_2\right)^{-1} = \mathbf{\Phi}_{2\mathbf{qp}}\mathbf{\Phi}_{2\mathbf{pp}}^{-1} \tag{34}$$

invariant. In particular, this enables us to perform Gaussian elimination (via column reduction) on the right-side columns of each cumulative subpropagator, separating orders of magnitude spanned by solutions of the ODE. This crucial advantage mitigates otherwise catastrophic numerical error in taking the matrix product of subpropagators, which may span many orders of magnitude ($\sim \mathcal{O}(10^{40})$, say).

Another advantage of the domain-decomposed integration of the state transition matrix is its suitability for integrating near singular rational surfaces. We note that subpropagators are invertible, and therefore satisfy

$$\mathbf{\Phi}_{21} = \mathbf{\Phi}_{12}^{-1}. \tag{35}$$

This feature admits a convenient reversibility of the direction of integration.

As is well known, the ideal MHD ELEs have regular singularities at rational surfaces, at the magnetic axis, and at the separatrix. While [2] approaches the challenge of integrating across such surfaces with asymptotic expansions, the state transition matrix approach affords a simplification.

As was previously noted in [10], the solutions to ELE are well-behaved when they are integrated *away* from the singular surfaces; a solution which asymptotically diverges in the forward direction of integration, *decays* in the reverse direction. Therefore, no numerical instability is created, and the integrated modes retain their linear independence.

As a result, the integration across the singular surface may be achieved by integrating backward from the singular surface, and taking the matrix inverse of the resulting subpropagator. This is then multiplied with the forward-integrated subpropagator on the right of the singular surface.

It is worth further emphasizing the freedom that subdividing the domain of integration affords. Its adaptability may render it useful for calculations outside the ideal MHD model, perhaps providing a convenient numerical technique to solve resistive and high toroidal mode number MHD stability problems, for example. A fine enough subdivision of the integration may also allow intervals to be integrated near singular surfaces without reversal of the direction of integration. That is, provided subdomains are chosen small enough to bound the growth of all modes, the integrated modes would remain well-behaved, and in particular maintain their linear independence.

IV. Parallel Implementation

A. Grid-packing algorithm

In our parallel adaptation of DCON, we integrate the ODE of Eq. (23) using the ZVODE [1] complex adapative integrator, and we parallelize the integration via OpenMP [11] by subdividing the interval between the magnetic axis and plasma edge. Increased computational cost of our

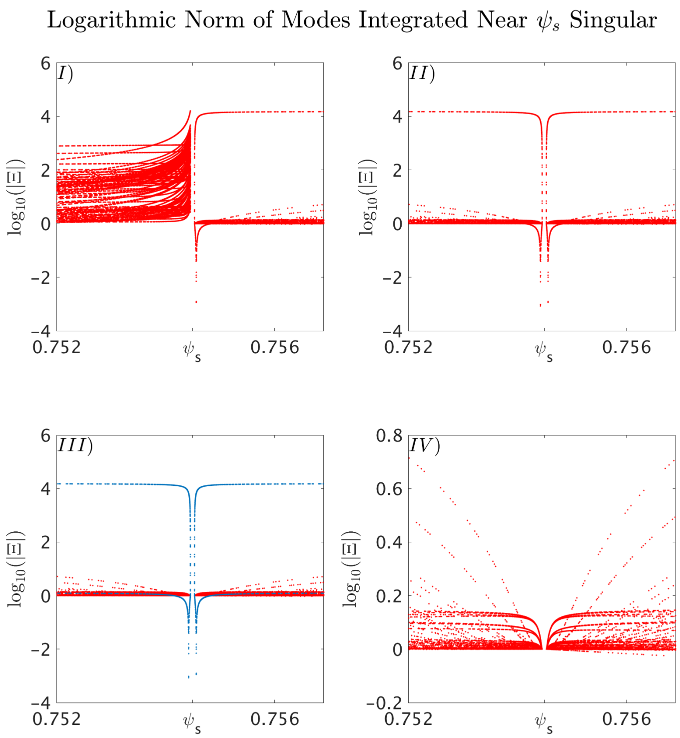

Fig. 1. I) The forward integration of modes without modification exhibits asymmetric behavior near the singular point. On approach to $\psi = \psi_s$, all modes are dominated by their projection along the asymptotically diverging singular mode and numerically lose their linear independence. II) A reversal of the integration to the left of the surface restores symmetry and linear independence to the solutions on both sides of the singularity. III) Highlighting the resonant modes $(m, m+M)$ at ψ_s—i.e., $q(\psi_s) = m/n$. IV) Integration reversal for $\psi < \psi_s$ *and* the exclusion of resonant modes.

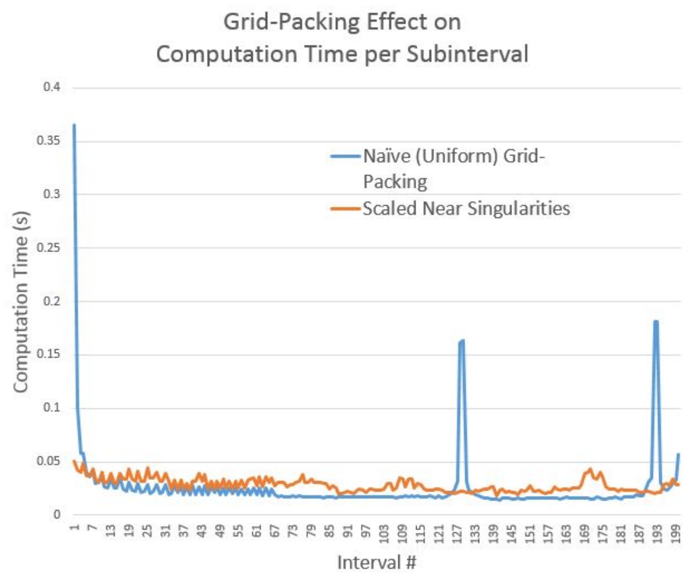

Fig. 2. Time of integration is seen to spike on intervals near the magnetic axis and singular surfaces for a naive, even-interval integration (blue). After applying the grid-packing scheme, the computation time per interval is substantially smoothed (orange).

ODE near ill-behaved surfaces (to wit, the magnetic axis and separatrix, and the singular surfaces between them), requires a careful division of integration intervals.

We find that a modification to a grid-packing algorithm suggested in [3] works adequately to load-balance the computation. In particular, we numerically fit a form factor

$$\mathrm{d}\tau(\psi) = \sum_i \frac{\alpha_i}{1 + \beta_i \left|\psi - \psi_{s i}\right|} \mathrm{d}\psi \tag{36}$$

to estimate the the time of integration near a point ψ, some distance from each nonanalytic surface $\psi_{s i}$. The coefficients (α_i, β_i) are determined by the surface type of $\psi_{s i}$ (axial, rational surface, or edge separatrix). The total time of integration over an interval $[\psi_1, \psi_2]$ is therefore estimated to be

$$\tau = \gamma_0 + \int_{\psi_1}^{\psi_2} \mathrm{d}\tau(\psi), \tag{37}$$

where γ_0 is fit to the unavoidable initialization time of the ZVODE integrator on each subinterval.

With such an approximation for the time of integration, the grid packing intervals are iteratively chosen as follows. Beginning with a set of intervals that divide the grid at each $\psi_{s i}$, interposing surfaces are chosen at each iteration to halve the largest remaining time interval, until the desired number of intervals is reached. The time expense of such a scheme for interval division is nearly free.

The importance of grid-packing is emphasized in Fig. 2. It is apparent that without an effective packing scheme, the benefit of parallelization would be quite limited; a single interval might otherwise require runtime comparable to runtimes of the entire parallelized code.

B. Performance and accuracy

As shown in Fig. 3, the eigenvalues of DCON's plasma response matrix are reproduced with high accuracy using our new parallel methodology. In this respect, domain decomposition via state transmission matrices serves as a faithful replacement for matched asymptotic expansions in solving DCON's singular ODE.

Even without a complete optimization of the parallel code, our initial implementation achieves a consistent runtime of ~ 490ms for an ITER-relevant EFIT equilibrium [8] with two rational surfaces. Given a projected ITER confinement time of $\tau_E \sim 5$s [5], such a runtime places our algorithm in the range of viability for an ideal MHD control system implementation at ITER. This result was achieved using 2.4 GHz Intel Broadwell processors, with 28 cores. The efficiency of our parallel implementation is depicted in Fig. 4.

One factor limiting the scalability of our approach is a tradeoff between the integration speedup of a finer subdivision of intervals, on the one hand, and the nontrivial time required to matrix-multiply subpropagators, on the other. Data for runtimes up to 20 threads project an optimal division of the integration into ~ 47 subdomains, ideally run with one core assigned per subdomain. With a limited threadcount, it has been optimal to divide the interval into

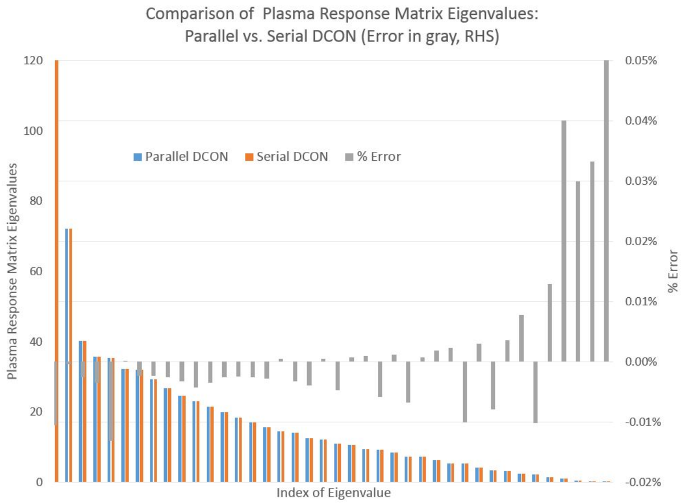

Fig. 3. Plasma response matrix eigenvalues are accurate to within 0.1% of serial DCON's.

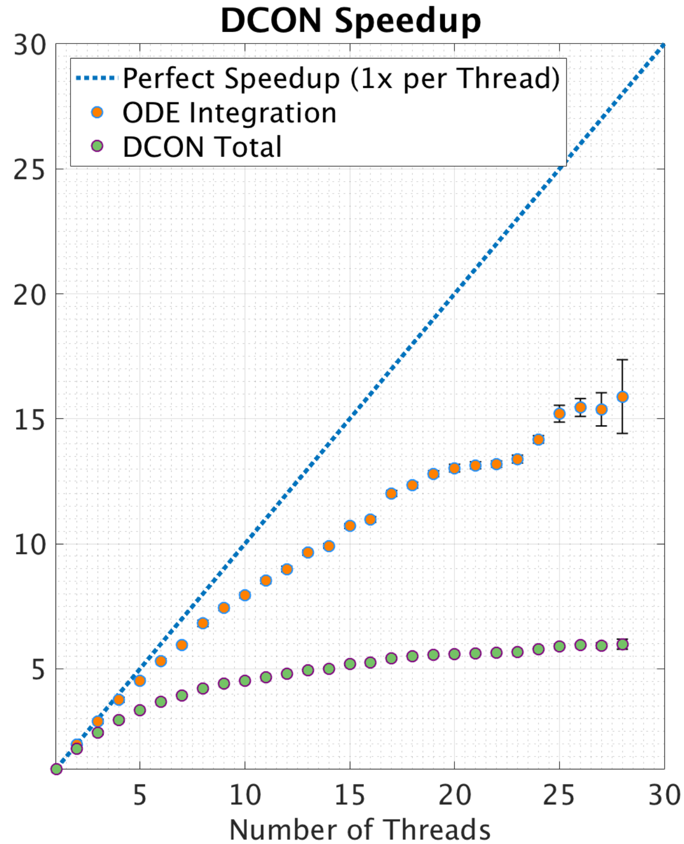

Fig. 4. The most computationally intensive part of DCON's stability calculation—the integration of the Euler-Lagrange ODE for the bulk-fluid perturbations—benefits from the addition of CPU's even up to 28 cores. Amdahl's law is seen to constrain the improvement achieved for the overall runtime.

40 subdomains.

V. Conclusion

We have demonstrated DCON's suitability to parallel computation, and achieved runtimes at a timescale appropriate for tokamak ideal MHD stability control. We have demonstrated the numerical advantage of using state transmission matrices to exploit DCON's linear Hamiltonian structure, and derived an alternate representation of the plasma response matrix as a solution to a Riccati differential equation.

We surmise that the vast literature on the solution of Riccati equations may be further exploited to simplify ideal MHD stability analyses. One might, for example, consider integrating the MRDE Eq. (20) itself to solve for the plasma response matrix. Although the singularities of this equation are just as virulent as the singularities of the ODE in Eq. (8)—(these nonanalytic features appear unavoidably in the coefficients of the MRDE themselves, after all)—there may yet be methods equivalent to asymptotic matching, represented in the reduced Riccati system, that prove useful.

The simplification that state transition matrices provide may also prove quite useful in a range of other numerical applications. This simple and seemingly overlooked feature of the linear ODE's present in ideal MHD stability analyses may be used to improve calculations in the plasma edge, or to analyze stability for resistive MHD equilibria.

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